A Note on the Orthogonal Basis of a Certain Full Symmetry Class of Tensors

C. Bessenrodt, M. R. Pournaki*, A. Reifegerste

Abstract

It is shown that the full symmetry class of tensors associated with the irreducible character $[2, 1^{n-2}]$ of S_n does not have an orthogonal basis consisting of decomposable symmetrized tensors.

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1 Introduction and Preliminaries

Let V be an m-unitary space. Let $\overset{n}{\otimes} V$ be the nth tensor power of V and write $v_1 \otimes \cdots \otimes v_n$ for the decomposable tensor product of the indicated vectors. To each permutation σ in S_n there corresponds a unique linear operator $P(\sigma)$: $\overset{n}{\otimes} V \to \overset{n}{\otimes} V$ determined by $P(\sigma)(v_1 \otimes \cdots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}$. Let G be a subgroup of S_n and let $\mathrm{Irr}(G)$ be the set of all the irreducible complex characters of G. It follows from the orthogonality relations for characters that

$$\left\{ T(G, \chi) : \overset{n}{\otimes} V \to \overset{n}{\otimes} V \,\middle|\, T(G, \chi) = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma) P(\sigma), \ \chi \in \mathrm{Irr}(G) \right\}$$

is a set of annihilating idempotents which sum to the identity. The image of $\overset{n}{\otimes} V$ under $T(G, \chi)$ is called the *symmetry class of tensors* associated with G and χ and it is denoted by $V_\chi^n(G)$. The image of $v_1 \otimes \cdots \otimes v_n$ under $T(G, \chi)$ is denoted by $v_1 * \ldots * v_n$ and it is called a *decomposable symmetrized tensor*.

The inner product on V induces an inner product on $\overset{n}{\otimes} V$ whose restriction to $V_\chi^n(G)$ satisfies

$$\langle u_1 * \ldots * u_n | v_1 * \ldots * v_n \rangle = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma) \prod_{i=1}^{n} \langle u_i | v_{\sigma(i)} \rangle. \qquad (1)$$

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Let Γ_m^n be the set of all sequences $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $1 \leq \alpha_i \leq m$. Then the group G acts on Γ_m^n by $\sigma \cdot \alpha = (\alpha_{\sigma^{-1}(1)}, \ldots, \alpha_{\sigma^{-1}(n)})$, where $\sigma \in G$ and $\alpha \in \Gamma_m^n$. Let $O(\alpha) = \{\sigma \cdot \alpha | \ \sigma \in G\}$ be the *orbit* of α, and G_α be its *stabilizer subgroup*, i.e., $G_\alpha = \{\sigma \in G | \ \sigma \cdot \alpha = \alpha\}$, and consider a system Δ of distinct representatives of the G-orbits on Γ_m^n.

Suppose $\{e_1, \ldots, e_m\}$ is an orthonormal basis of V. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \Gamma_m^n$, denote by e_α^* the decomposable symmetrized tensor $e_{\alpha_1} * \ldots * e_{\alpha_n}$. Then, by (1), one can easily obtain that for each $\alpha, \beta \in \Gamma_m^n$,

$$\langle e_\alpha^* | e_\beta^* \rangle = \begin{cases} \dfrac{\chi(1)}{|G|} \displaystyle\sum_{\sigma \in G_\beta} \chi(\sigma\tau^{-1}) & \text{if } \alpha = \tau \cdot \beta \text{ for some } \tau \in G, \\ 0 & \text{if } O(\alpha) \neq O(\beta). \end{cases} \tag{2}$$

For $\alpha \in \Delta$, $V_\alpha^* = \langle e_{\sigma \cdot \alpha}^* | \ \sigma \in G \rangle$ is called the *orbital subspace* of $V_\chi^n(G)$, and we can easily prove that

$$V_\chi^n(G) = \bigoplus_{\alpha \in \Delta} V_\alpha^*. \tag{3}$$

Note that it is possible for some $\alpha \in \Delta$ to have $V_\alpha^* = 0$. But Freese [3] proved

$$\dim V_\alpha^* = \frac{\chi(1)}{|G_\alpha|} \sum_{\sigma \in G_\alpha} \chi(\sigma), \tag{4}$$

therefore, if we set

$$\overline{\Delta} = \Big\{ \alpha \in \Delta | \sum_{\sigma \in G_\alpha} \chi(\sigma) \neq 0 \Big\},$$

then by (3) we obtain

$$V_\chi^n(G) = \bigoplus_{\alpha \in \overline{\Delta}} V_\alpha^*. \tag{5}$$

Of course we define the right-hand side of (5) to be 0, if $\overline{\Delta} = \emptyset$.

Let W be a subspace of $V_\chi^n(G)$. An orthogonal basis of W of the form

$$\left\{ e_\alpha^* | \ \alpha \in S \right\},$$

where S is a subset of Γ_m^n, is called an *O-basis* of W.

Symmetry classes of tensors associated with subgroups of symmetric groups and their irreducible characters have been studied for a long time. Several papers are devoted to the investigation of the non-vanishing and existence of an O-basis for $V_\chi^n(G)$, see for example [1, 2, 5, 6, 8, 9]. In recent years the non-vanishing problem of decomposable symmetrized tensors and so the non-vanishing problem of $V_\chi^n(G)$, in the case $G = S_n$, have been studied by several authors, see for example [4, 7], and the non-vanishing problem in this case has been completely solved (see [4]). But even for $G = S_n$, no reasonable result for the structure of O-basis of an $V_\chi^n(G)$ is available. In this note we consider $G = S_n$ and investigate the existence of an O-basis of $V_\chi^n(S_n)$, for a special irreducible character of S_n.

2 Main Result

Let n be a positive integer. A *partition* $\lambda = (\lambda_1, \dots, \lambda_l)$ of n is a weakly decreasing sequence $\lambda_1 \geq \dots \geq \lambda_l > 0$ of integers with $\sum_{i=1}^{l} \lambda_i = n$, for short we write $\lambda \vdash n$. The number l is called the *length* of λ, denoted by $l(\lambda)$. We often gather together equal parts of a partition and write, for example, $(5^2, 3^3)$ for $(5, 5, 3, 3, 3)$. If $\lambda = (\lambda_1, \dots, \lambda_l) \vdash n$, then $\lambda' = (\lambda'_1, \dots, \lambda'_s)$, defined by

$$\lambda'_i = \big|\{1 \leq j \leq l |\ \lambda_j \geq i\}\big|,$$

is a partition of n called the *partition conjugate* to λ.

Frobenius obtained in 1900 an explicit classification of the irreducible complex characters of S_n; they are naturally labelled by partitions of n. We denote the irreducible complex character labelled by the partition λ by $[\lambda]$, so the set of all irreducible complex characters of S_n is $\mathrm{Irr}(S_n) = \{[\lambda] |\ \lambda \vdash n\}$.

Let now V be an m-unitary space. The symmetry class of tensors associated with S_n and $[\lambda]$, where $\lambda \vdash n$, is called *full symmetry class of tensors* associated with λ and for short denoted by V_λ^n. Holmes [5] proved that if $m, n \geq 3$, then $V_{(n-1,1)}^n$ is non-zero and does not have an O-basis. In this note we consider the special case $\lambda = (2, 1^{n-2})$, that is the partition conjugate to $(n-1, 1)$; we prove the analogue of Holmes' result for the space V_λ^n.

The following result was already proved in [7].

Proposition 2.1 *Let V be an m-unitary space. Let λ be a partition of n. Then the full symmetry class of tensors associated with λ, i.e., V_λ^n, is non-zero if and only if $m \geq l(\lambda)$. In particular, if $m \geq n-1$, then $V_\lambda^n \neq 0$ for all $\lambda \neq (1^n)$.*

We are now ready to state our main result. Note that by Proposition 2.1, V_λ^n is non-zero for $\lambda = (2, 1^{n-2})$ if and only if $m \geq n - 1$.

Main Theorem *Let $n \geq 3$ and consider $\lambda = (2, 1^{n-2})$. Let V be an m-unitary* *space, $m \geq n-1$. Then the full symmetry class of tensors associated with λ, i.e., V_λ^n, does not have an O-basis.*

3 Proof of the Main Theorem

For the proof of the Main Theorem we need a combinatorial result on permutations. We denote by π the natural permutation character of S_n, i.e., for $\sigma \in S_n$, $\pi(\sigma)$ is the number of fixed points of σ.

Lemma 3.1 *Set $F = \{\sigma \in S_n | \ \pi(\sigma) = \pi(\sigma\,(1\,2))\}$. Then we have $F = \{\sigma \in S_n | \ \{\sigma(1), \sigma(2)\} \cap \{1,2\} = \emptyset\}$.*

Proof. Let $\sigma \in S_n$. There are four possibilities. If $\{\sigma(1), \sigma(2)\} \cap \{1,2\} = \emptyset$, then σ and $\sigma\,(1\,2)$ have exactly the same fixed points. If $\{\sigma(1), \sigma(2)\} = \{1,2\}$, then the number of fixed points of σ and $\sigma\,(1\,2)$ differ by 2. Now assume that $\{\sigma(1), \sigma(2)\} = \{1, a\}$, with $a \neq 2$. Then in one-line notation the permutations σ and $\sigma\,(1\,2)$ are (not necessarily in that order) $1\,a\,\sigma(3)\ldots\sigma(n)$ and $a\,1\,\sigma(3)\ldots\sigma(n)$, hence their fixed point numbers differ by 1. The case where $\{\sigma(1), \sigma(2)\} = \{2, a\}$, with $a \neq 1$ is similar. This proves the claim. $\square$

Lemma 3.2 *Let $F = \{\sigma \in S_n | \ \pi(\sigma) = \pi(\sigma\,(1\,2))\}$, and let $\sigma_1, \ldots, \sigma_k$ be distinct permutations of S_n such that $\sigma_i\sigma_j^{-1} \in F$ for all $i \neq j$. Then $k \leq [n/2]$.*

Proof. Set $t_r = \sigma_1\sigma_{r+1}^{-1}$ for $r = 0, \ldots, k-1$, so $t_0 = \mathrm{id}$ and $t_1, \ldots, t_{k-1} \in F$. Then $t_r^{-1}t_s = \sigma_{r+1}\sigma_1^{-1}\sigma_1\sigma_{s+1}^{-1} = \sigma_{r+1}\sigma_{s+1}^{-1} \in F$ for all $r \neq s$, $r, s \in \{1, \ldots, k-1\}$. Hence by Lemma 3.1, $\{t_r^{-1}t_s(1), t_r^{-1}t_s(2)\} \cap \{1,2\} = \emptyset$, or equivalently, $\{t_s(1), t_s(2)\} \cap \{t_r(1), t_r(2)\} = \emptyset$ for all $r \neq s$, $r, s \in \{1, \ldots, k-1\}$. Thus

$$2k = \left|\{t_r(j) | \ \ r = 0, \ldots, k-1, \ j \in \{1,2\}\}\right| \leq n$$

and so $k \leq [n/2]$. $\square$

Remark 3.3 The bound $[n/2]$ in Lemma 3.2 is sharp, and the proof shows how to construct such sets of permutations. In particular, we obtain such a set by setting $\sigma_1 = \mathrm{id}$ and (in cycle notation) $\sigma_j = (1\ 2j-1)\,(2\ 2j)$ for $j = 2, \ldots, [n/2]$.

We are now ready to prove the Main Theorem. Let $\{e_1, \ldots, e_m\}$ be an orthonormal basis of V. Assume that V_λ^n, $\lambda = (2, 1^{n-2})$, has an O-basis. Put $\gamma = (1, 1, 2, 3, \ldots, n-1)$, then $n \geq 3$ and $m \geq n-1$ implies that $\gamma \in \Gamma_m^n$. Consider the action of S_n on Γ_m^n and choose Δ such that $\gamma \in \Delta$. It is easy to see that the stabilizer subgroup of γ is equal to $(S_n)_\gamma = \{(1),(12)\}$. Therefore

$$\sum_{\sigma\in(S_n)_\gamma}[\lambda](\sigma) = (n-1)+(-1)(n-3) = 2,$$

so it is non-zero and we obtain $\gamma\in\overline{\Delta}$. $\big($In fact, this sum is non-zero for any partition $\lambda\neq(1^n)$, so $\gamma\in\overline{\Delta}$ for any $\lambda\neq(1^n)$.$\big)$ Now, by (5), we can decompose V_λ^n into the orthogonal direct sum of orbital subspaces indexed by $\overline{\Delta}$. Since V_λ^n has an O-basis, and $\gamma\in\overline{\Delta}$, the orthogonality of this decomposition implies that V_γ^* has an O-basis. But, by (4), we have

$$\dim V_\gamma^* = \frac{n-1}{2}\cdot 2 = n-1,$$

so we can assume $\{e^*_{g_1\cdot\gamma},\ldots,e^*_{g_{n-1}\cdot\gamma}\}$ is an O-basis for V_γ^*. Therefore for each $i\neq j$, $1\leq i,j\leq n-1$, we have $\langle e^*_{g_i\cdot\gamma}|e^*_{g_j\cdot\gamma}\rangle = 0$. On the other hand if $\alpha = g_i\cdot\gamma$ and $\beta = g_j\cdot\gamma$, then $g_ig_j^{-1}\cdot\beta=\alpha$, so if we set $\tau = g_ig_j^{-1}$ and use (2), then we obtain

$$\begin{aligned}\langle e^*_{g_i\cdot\gamma}|e^*_{g_j\cdot\gamma}\rangle &= \frac{[\lambda](1)}{|S_n|}\sum_{\sigma\in(S_n)_\gamma}[\lambda](g_i^{-1}g_j\sigma)\\ &= \frac{n-1}{n!}\Big(\varepsilon(g_ig_j^{-1})\big(\pi(g_ig_j^{-1})-1\big)+\varepsilon\big(g_ig_j^{-1}(12)\big)\big(\pi\big(g_ig_j^{-1}(12)\big)-1\big)\Big)\\ &= \frac{n-1}{n!}\,\varepsilon(g_ig_j^{-1})\Big(\pi(g_ig_j^{-1})-\pi\big(g_ig_j^{-1}(12)\big)\Big),\end{aligned}$$

where ε is the sign character. Now the condition $\langle e^*_{g_i\cdot\gamma}|e^*_{g_j\cdot\gamma}\rangle = 0$, for each $i\neq j$, $1\leq i,j\leq n-1$, implies that for such i,j we have $\pi(g_ig_j^{-1}) = \pi\Big(g_ig_j^{-1}(12)\Big)$. Hence we deduce from Lemma 3.2 that $n-1\leq[n/2]$. Since $n\geq 3$, the last inequality is a contradiction and thus V_λ^n, $\lambda=(2,1^{n-2})$, does not have an O-basis. □

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The Authors' Addresses

C. Bessenrodt, Institut für Mathematik, Universität Hannover, Welfengarten 1, D-30167 Hannover, Germany.

E-mail address: `bessen@math.uni-hannover.de`

M. R. Pournaki, School of Mathematics, Institute for Studies in Theoretical Physics and Mathematics, P.O. Box 19395-5746, Tehran, Iran.

E-mail address: `pournaki@ipm.ir`

A. Reifegerste, Institut für Mathematik, Universität Hannover, Welfengarten 1, D-30167 Hannover, Germany.

E-mail address: `reifegerste@math.uni-hannover.de`